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Painlevé Analysis for a Nonlinear Schrödinger Equation in Three Dimensions

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A Painlevé analysis is performed for the nonlinear Schrödinger equation in (2+1) dimensions following the methodology of Weiss *et al.* simplified in the sense of Kruskal. At least for one branch it is found that the required number of arbitrary functions (as demanded by the Cauchy-Kovalevskaya theorem) exists, signalling complete integrability.

1. INTRODUCTION

The question of the integrability of nonlinear partial differential equations has gained a tremendous boost in the last decade due to the relationship that exists between the chaotic behavior of some systems and nonintegrability (Ramai *et al.*, 1986). Of late various classes of equations have been analyzed on the basis of the Painlevé test (Weiss *et al.*, 1983; Weiss, 1983, 1984a,b). Although the Painlevé test is not a necessary and sufficient condition for integrability of a p.d.e., it has worked in many known and important situations. Here we report one such analysis for the nonlinear Schrödinger equation (NLSE) two space (+ one time) dimensions (Mukherjee and Roy Chowdhury, 1985). The only other equation in (2+1) dimensions whose Painlevé analysis has been done is the KP equation (Chudnovsky *et al.*, 1983). It is found that the 2D NLSE is completely integrable in the sense of the Painlevé test, satisfying all the requirements of the Cauchy-Kovalevskaya theorem.

The 2D NLSE is written as

$$\begin{aligned} p_t &= Ap_{xx} - Bp_{yy} + (r-s)p \\ q_t &= Bq_{yy} - Aq_{xx} + (s-r)q \\ r_x &= 2B(pq)_y \\ s_y &= 2A(pq)_x \end{aligned} \tag{1}$$

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2. LEADING ORDER ANALYSIS

To estimate the leading singularity we assume

$$p \sim p_0\phi^{\alpha}; \qquad q \sim q_0\phi^{\beta}; \qquad r \sim r_0\phi^{\gamma}; \qquad s \sim s_0\phi^{\delta} \tag{2}$$

where $\phi = x - f(y, t)$ and (p_0, q_0, r_0, s_0) are all functions of (y, t). This form of assumption was used by Jimbo *et al.* (1982) and Goldstein and Infeld (1984), and was initially suggested by M. Kruskal (private communication), whereas, according to the original prescription of Weiss *et al.* (1983), ϕ is a general function of (x, y, t). If we substitute (2) in (1) and consider that these exponents will be all negative integers, then we get $\alpha = -1$, $\beta = -1$, $\gamma = -2$, $\delta = -2$, whereas the leading equations are

$$\begin{aligned} p_{0t}\phi^{\alpha} - \alpha p_0\phi^{\alpha-1}f_t \\ &= Ap_0\alpha(\alpha-1)\phi^{\alpha-2} - Bp_{0yy}\phi^{\alpha} + 2Bp_{0y}\alpha\phi^{\alpha-1}f_y \\ &\quad - Bp_0\alpha(\alpha-1)\phi^{\alpha-2}f_y^2 + Bp_0\alpha\phi^{\alpha-1}f_{yy} + r_0p_0\phi^{\alpha+\gamma} - s_0p_0\phi^{\alpha+\delta} \end{aligned} \tag{3a}$$

$$\begin{aligned} q_{0t}\phi^{\beta} - \beta q_0\phi^{\beta-1}f_t \\ &= Bq_{0yy}\phi^{\beta} - 2Bq_{0y}\beta\phi^{\beta-1}f_y + Bq_0\beta(\beta-1)\phi^{\beta-2}f_y^2 \\ &\quad - Bq_0\beta\phi^{\beta-1}f_{yy} - Aq_0\beta(\beta-1)\phi^{\beta-2} + q_0s_0\phi^{\beta+\delta} - q_0r_0\phi^{\beta+\gamma} \end{aligned} \tag{3b}$$

$$r_0\gamma\phi^{\gamma-1} = 2B(p_0q_0)_y\phi^{\alpha+\beta} - 2B(p_0q_0)(\alpha+\beta)\phi^{\alpha+\beta-1}f_y \tag{3c}$$

$$s_{0y}\phi^{\delta} - s_0\delta\phi^{\delta-1}f_y = 2A(p_0q_0)(\alpha+\beta)^{\alpha+\beta-1} \tag{3d}$$

Most singular terms, when equated leads to

$$p_0q_0 = -f_y, \qquad r_0 = 2Bf_y^2, \qquad s_0 = 2A \tag{4}$$

so that one of (p_0, q_0) is arbitrary. Note that we may have other possibilities for the exponents $(\alpha, \beta, \gamma, \delta)$, for example, $\delta = \gamma = -1$, but $\alpha + \beta = -1$. But in these cases the leading equations are almost decoupled:

$$Ap_{xx} - Bp_{yy} = 0, \qquad Bq_{yy} - Aq_{xx} = 0, \qquad r_x = 2B(pq)_y, \qquad s_y = 2A(pq)_x \tag{5}$$

Since all the exponents are nonnegative integers (as $\alpha + \beta = -1$) and the leading equations are decoupled, we do not consider these cases.

Another possibility is $\delta = -2$, $\gamma = -2$, $\alpha + \beta = -2$, $\alpha \neq -1$, $\beta \neq -1$. In this case we may have several fractional values of (α, β) so that $\alpha + \beta = -2$. But in these cases the basic assumption of a moving pole structure is violated. Some authors speak of such situations as having the weak Painlevé property. In this case also we get completely decoupled leading order equations.

Under the above circumstances we proceed to search for the resonance positions only in the first case.

2. RESONANCE POSITIONS

To obtain the system matrix giving yhe resonance positions, we now set

$$
\begin{aligned}
p &= \sum_{j=0}^{\infty} p_j \phi^{j-1}; \qquad q = \sum_{j=0}^{\infty} q_j \phi^{j-1} \\
r &= \sum_{j=0}^{\infty} r_j \phi^{j-2}; \qquad s = \sum_{j=0}^{\infty} s_j \phi^{j-2}
\end{aligned}
\tag{6}
$$

Equating coefficients of ϕ^{m-3}, we get

$$
\begin{bmatrix}
A(m-1)(m-2)+r_0-s_0 - B(m-1)(m-2)f_y^2 & 0 & p_0 & -p_0 \\
0 & -A(m-1)(m-2)-r_0+s_0 + B(m-1)(m-2)f_y^2 & -q_0 & q_0 \\
2Bf_y q_0(m-2) & 2Bf_y p_0(m-2) & m-2 & 0 \\
2Aq_0(m-2) & 2Ap_0(m-2) & 0 & (m-2)f_y
\end{bmatrix}
$$

$$
\times \begin{bmatrix} p_m \\ q_m \\ r_m \\ s_m \end{bmatrix}
= \begin{bmatrix}
X \\
Y \\
2B \sum_{n=0}^{m-1} (p_{m-n-1} q_n)_y - 2Bf_y \sum_{n=1}^{m-1} p_{m-n} q_n (m-2) \\
s_{m-1,y} - 2A \sum_{n=1}^{m-1} p_{m-n} q_n (m-2)
\end{bmatrix}
\tag{7}
$$

where the expressions X and Y are given as

$$
\begin{aligned}
X &= p_{m-2,t} - p_{m-1}(m-2)f_t + Bp_{(m-2)yy} - 2Bp_{(m-1)y} \\
&\quad \times (m-2)f_y - Bp_{m-1}(m-2)f_{yy} + \sum_{n=1}^{m-1} r_{m-n}p_n - \sum_{n=1}^{m-1} s_{m-n}p_n, \\
Y &= q_{m-2t} - q_{m-t}(m-2)f_t - Bq_{m-2yy} + 2Bq_{m-1y}(m-2)f_y \\
&\quad + Bq_{m-1}(m-2)f_{yy} - \sum_{n=1}^{m-1} s_{m-n}q_n + \sum_{n=1}^{m-1} r_{m-n}q_n
\end{aligned}
\tag{8}
$$

Resonance positions are those values at $m = r$ for which the determinant of the system matrix [occurring on the left-hand side of (7)] vanishes. It is found that

$$
\det[\,\cdot\,] = (m+1)m(m-2)^2(m-3)(m-4)
$$

so that we get resonances at

$$
r = -1, 0, 2, 2, 3, 4 \tag{9}
$$

3. COEFFICIENTS OF EXPANSION AT THE RESONANCE POSITION

Case a. $r=-1$. This corresponds to the arbitrariness of

$$\phi = x - f(y, t) \tag{10a}$$

Case b. $r=0$. This represents the arbitrariness of p_0 or q_0 [see equation (4)].

Case c. $r=1$. We get from (7)

$$\begin{aligned} p_1 &= \frac{1}{r_0 - s_0}(-Bp_0 f_{yy} + q_0 f_t + 2Bp_{0y} f_y) \\ q_1 &= \frac{1}{r_0 - s_0}(-Bq_0 f_{yy} - q_0 f_t + 2Bq_0 f_y) \\ r_1 &= 2Bf_{yy}, \qquad s_1 = 0 \end{aligned} \tag{10b}$$

Case d. $r=+2$ (double resonance). In this case

$$\begin{aligned} p_2 &= \frac{1}{r_0 - s_0}[-p_0(r_2 - s_2) + p_{0t} + Bp_{0yy} - r_1 s_1] \\ q_2 &= \frac{1}{r_0 - s_0}[-q_0(r_2 - s_2) - q_{0t} + Bq_{0yy} - r_1 s_1] \\ r_2 &= s_2 \quad \text{arbitrary} \end{aligned} \tag{11}$$

subject to the compatibility condition

$$(p_0 q_1 + p_1 q_0)_y = 0, \qquad s_{1y} = 0$$

which are obtained from the third and fourth rows of (7). But it is not difficult to verify that these conditions are identically satisfied by those obtained in equations (4), (10a), and (10b).

Case e. $r=3$. Here we get

$$\begin{aligned} r_3 &= 2B(p_2 q_0 + p_1 q_1 + p_0 q_2)_y - 2Bf_y(p_3 q_0 + p_2 q_1 + p_1 q_2 + p_0 q_3) \\ s_3 &= \frac{1}{f_y}[s_{2y} - 2A(p_3 q_0 + p_2 q_1 + p_1 q_2 + p_0 q_3)] \\ (q_0 p_3 + p_0 q_3) &= \frac{f_y}{r_0 - s_0}\left\{ -\frac{1}{f_y}(r_0 - s_0)(p_2 q_1 + q_2 p_1) \right. \\ &\quad + 2B(p_2 q_0 + p_1 q_1 + p_0 q_0)_y - \frac{s_{2y}}{f_y} \\ &\quad - \frac{1}{p_0}[p_{1t} - p_2 f_t + Bp_{1yy} - 2Bp_{2y} f_y \\ &\quad \left. - Bp_2 f_{yy} + s_2 q_1 - r_2 q_1 - r_1 p_2] \right\} \end{aligned} \tag{12}$$

Therefore one of p_3, q_3 can be assumed to be arbitrary, subject to the following compatibility condition:

$$
\begin{aligned}
&2B_f(p_0q_{2y}-q_0p_2)+3Bf_{yy}(p_0q_2-q_0p_2)\\
&\quad-(p_0q_2+q_0p_2)f_y+(p_0q_1-q_0p_1)(r_2-s_2)+(q_0p_{1t}+p_0q_{1t})\\
&\quad+B(q_0p_{1yy}-p_0q_{1yy})+s_1(q_0p_2-p_0q_2)=0
\end{aligned}
\tag{13}
$$

But by substituting the values of (p_0, q_0), (p_2, q_2), etc., it can be seen that (13) is identically satisfied.

Case f. $r=4$. At the last resonance position the recurrence relation leads to

$$
\begin{aligned}
r_4&=\tfrac{1}{2}[-4Bf_y(q_0p_4+p_0q_4)+2B(p_3q_0+p_2q_1+p_1q_2+p_0q_3)_y\\
&\quad-4Bf_y(p_3q_1+p_2q_2+p_1q_3)]\\
s_4&=\frac{1}{2f_y}[-4A(q_0p_4+p_0q_4)+s_{3y}-4A(q_3p_1+p_2q_2+p_3q_1)]\\
(q_0p_4-p_0q_4)&=+\frac{1}{s_0-r_0}\{f_y[B(p_3q_0+p_2q_1+p_1q_2+q_0p_3)_y\\
&\quad-2Bf_y(p_3q_1+p_2q_2+p_1q_3)]\\
&\quad-[\tfrac{1}{2}s_{3y}-2A(p_3q_1+p_2q_2+p_1q_3)]\}
\end{aligned}
\tag{14}
$$

subject to a compatibility condition that comes from the right-hand side of the recursion relation (7). The said condition is

$$
\begin{aligned}
&(p_0q_{2t}-q_0p_{2t})-2(p_0q_3-q_0p_3)f_t-B(p_0q_{2yy}+q_0p_{2yy})\\
&\quad+4B(p_0q_{3y}+q_0p_{3y})f_y+2B(q_0p_3+p_0q_3)f_{yy}+r_3(p_0q_1+p_1q_0)\\
&\quad+r_2(p_0q_2+q_0p_2)+r_1(p_0q_3+p_3q_0)-s_3(p_0q_1+q_0p_1)\\
&\quad-s_2(p_0q_2+p_2q_0)-s_1(p_0q_3+q_0p_3)-2\{f_y[B(p_3q_0+p_2q_1\\
&\quad+p_1q_2+p_0q_3)_y-2Bf_y(p_3q_1+p_2q_2+p_1q_3)]\\
&\quad-[\tfrac{1}{2}s_{3y}-2A(p_3q_1+p_2q_2+p_1q_3)]\}=0
\end{aligned}
\tag{15}
$$

It is really amusing to note that such a complicated-looking equation is also identically satisfied by the coefficients determined previously.

4. CONCLUSION

The above analysis shows that our equation has six resonances at $r=-1$, 0, 2, 2, 3, 4 and we can have six arbitrary coefficients at these resonances positions [including $\phi(x, y, t)=x-f(y, t)$]. So the Cauchy-Kovalevskaya theorem immediately suggests that our equation does conform to the Painlevé criterion of complete integrability.

At this point it might not be out of place to note our Painlevé analysis cannot be used to deduce the Lax pair because as, in the case of a coupled system, there is still no concrete method to deduce the Lax pair for coupled nonlinear equations.

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